Geometric inequalities involving a,b,c,r,R
https://www.linkedin.com/groups/8313943/8313943-6359650907478597636
In a ΔABC, let a,b,c be its side lengths
R - its circumradius and r - its inradius, prove that
(**a**) $1/a+1/b+1/c \le \sqrt{3}\ /(2r)$
(**b**) $1/a+1/b+1/c \le (1/\sqrt{3})(1/r+1/R)$
(**c****) $11\sqrt{3}/(5R+12r) \le 1/a+1/b+1/c \le (1/\sqrt{3})(5/(4R)+7/(8r))$.
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Let $x := s-a, y := s-b, z := s-c$ and $p := xy+yz+zx, q := xyz.$ Also assume $s = 1$ (due to homogeneity of inequalities. Then $x+y+z = 1, x,y,z > 0, a = 1-x, b = 1-y,$ $c = 1-z,\ ab+bc+ca = 1+p, abc = p-q, \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{1+p}{p-q}, r = \sqrt{q}, R = \frac{p-q}{4\sqrt{q}}$

and noting that $q = xyz(x+y+z) \le \frac{(xy+yz+zx)^2}{3} = \frac{p^2}{3}$ we can to proceed to the proof of inequalities (**a**),(**b**),(**c**) in p,q notation:
Inequality in (**a**) becomes $\frac{1+p}{p-q} \le \frac{\sqrt{3}}{2\sqrt{q}} \Leftrightarrow 2(1+p) \le \frac{\sqrt{3}(p-q)}{\sqrt{q}}$.
Since $\frac{p-q}{\sqrt{q}}$ as function of q decrease in $(0,p^2/3]$ then $\frac{\sqrt{3}(p-q)}{\sqrt{q}} - 2(1+p) \ge$
$\frac{\sqrt{3}(p-p^2/3)}{p/\sqrt{3}} - 2(1+p) = 1-3p \ge 0;$
Inequality in (**b**) becomes $\frac{1+p}{p-q} \le \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{q}}+\frac{4\sqrt{q}}{p-q}\right) \Leftrightarrow 1+p \le \frac{1}{\sqrt{3}}\left(\frac{p}{\sqrt{q}}+3\sqrt{q}\right)$.
Since $\frac{p}{\sqrt{q}}+3\sqrt{q}$ as function of q decrease in $(0,p^2/3]$ ($\frac{p}{t}+3t$ decrease for $t \in (0,\sqrt{p/3}\,]$

and $\sqrt{q} \le p/\sqrt{3} < \sqrt{p/3}$ because $p < 1$) then $\frac{1}{\sqrt{3}}\left(\frac{p}{\sqrt{q}}+3\sqrt{q}\right) - (1+p) \ge$
$\frac{1}{\sqrt{3}}\left(\frac{p}{p/\sqrt{3}}+3p/\sqrt{3}\right) - (1+p) = 0.$
Remark.
Note that $\frac{1}{\sqrt{3}}\left(\frac{1}{r}+\frac{1}{R}\right) \le \frac{\sqrt{3}}{2r}$ because $R \ge 2r$ (Euler Inequality) and
$\frac{\sqrt{3}}{2r} - \frac{1}{\sqrt{3}}\left(\frac{1}{r}+\frac{1}{R}\right) = \frac{\sqrt{3}(R-2r)}{6Rr}$. Thus, it was sufficient to prove inequality (**b**).
Also note that $\frac{1}{\sqrt{3}}\left(\frac{1}{r}+\frac{1}{R}\right) \ge \frac{1}{\sqrt{3}}\left(\frac{5}{4R}+\frac{7}{8r}\right) \Leftrightarrow \frac{1}{r}+\frac{1}{R} - \left(\frac{5}{4R}+\frac{7}{8r}\right) \ge 0 \Leftrightarrow$
$\frac{1}{8}\frac{R-2r}{Rr} \ge 0.$
That is $\frac{1}{\sqrt{3}}\left(\frac{5}{4R}+\frac{7}{8r}\right) \le (\frac{1}{\sqrt{3}}\left(\frac{1}{r}+\frac{1}{R}\right) \le \frac{\sqrt{3}}{2r}$.
Well known inequality (Soltan and Meydman) $\frac{2(5R-r}{3\sqrt{3}R^2} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le \frac{(R+r)^2}{3\sqrt{3}Rr^2}$

but $\dfrac{(R+r)^2}{3\sqrt{3}\,Rr^2} \geq \dfrac{1}{\sqrt{3}}(1/r+1/R)$ because $\dfrac{(R+r)^2}{3\sqrt{3}\,Rr^2} - \dfrac{1}{\sqrt{3}}\left(\dfrac{1}{r}+\dfrac{1}{R}\right) = \dfrac{(R-2r)(R+r)}{3\sqrt{3}\,Rr^2} \geq 0.$

Since $\dfrac{1}{\sqrt{3}}\left(\dfrac{5}{4R}+\dfrac{7}{8r}\right) \leq \dfrac{1}{\sqrt{3}}\left(\dfrac{1}{r}+\dfrac{1}{R}\right) \leq \dfrac{\sqrt{3}}{2r}$ then proof of inequality

(**CR**) $\quad \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} \leq \dfrac{1}{\sqrt{3}}\left(\dfrac{5}{4R}+\dfrac{7}{8r}\right)$ be at the same time proof of inequalities in (**a**) and (**b**).

But proof of inequality (**CR**) is much more difficult then two others.

Proof.

In p,q notation inequality (**CR**) becomes $\dfrac{1+p}{p-q} \leq \dfrac{1}{\sqrt{3}}\left(\dfrac{5\cdot 4\sqrt{q}}{4(p-q)}+\dfrac{7}{8\sqrt{q}}\right) \Leftrightarrow$

$\dfrac{1+p}{p-q} \leq \dfrac{7p+33q}{8\sqrt{3}\,\sqrt{q}\,(p-q)} \Leftrightarrow 1+p \leq \dfrac{1}{8\sqrt{3}}\left(33\sqrt{q}+\dfrac{7p}{\sqrt{q}}\right) \Leftrightarrow$

(**1**) $\quad (1+p)^2 \leq \dfrac{77p}{32}+\dfrac{1}{192}\left(1089q+\dfrac{49p^2}{q}\right).$

Note that function $1089t+\dfrac{49p^2}{t}$ decrease in $\left(0,\dfrac{7p}{33}\right]$ and $\dfrac{p^2}{3} < \dfrac{7p}{33} \Leftrightarrow p < \dfrac{7}{11}.$

Thus $1089q+\dfrac{49p^2}{q}$ as function of q decrease in $\left(0,\dfrac{p^2}{3}\right].$

But unfortunately $\dfrac{p^2}{3}$ as upper bound for q isn't good enough for proof of inequality (**1**) and even more sharp inequality $q \leq \dfrac{p^2}{4-3p}$ can't provide proof of (**1**).

In that situation remains only way, that is to use the best upper bound for q which can give a criterion for the solvability of the Vieta's system of equations $a+b+c=1, ab+bc+ca=p, abc=q$ in real $a,b,c,$ namely inequality

(**2**) $\quad 27q^2-2(9p-2)q+4p^3-p^2 \leq 0.$

This inequality solvable in real q iff $p \leq \dfrac{1}{3}$ and being solved with respect to q in particular give us attainable upper bound for $q,$ namely $q \leq q_* := \dfrac{9p-2+2(1-3p)\sqrt{1-3p}}{27}.$

Denoting $t := \sqrt{1-3p}$ we obtain $p = \dfrac{1-t^2}{3}$ and $q_* = \dfrac{(1+2t)(1-t)^2}{27},$

where $0 \leq t < 1 \Leftrightarrow 0 < p \leq \dfrac{1}{3}.$

Then

$$\dfrac{77p}{32}+\dfrac{1}{192}\left(1089q+\dfrac{49p^2}{q}\right)-(1+p)^2 \geq \dfrac{77p}{32}+\dfrac{1}{192}\left(1089q_*+\dfrac{49p^2}{q_*}\right)-(1+p)^2 =$$

$$\dfrac{77}{32}\cdot\dfrac{1-t^2}{3}+\dfrac{1}{192}\left(1089\cdot\dfrac{(1+2t)(1-t)^2}{27}+\dfrac{49(1-t^2)^2}{9}\cdot\dfrac{27}{(1+2t)(1-t)^2}\right)-\dfrac{(4-t^2)^2}{9} =$$

$$\dfrac{77}{32}\cdot\dfrac{1-t^2}{3}+\dfrac{1}{192}\left(1089\cdot\dfrac{(1+2t)(1-t)^2}{27}+\dfrac{49(1-t^2)^2}{9}\cdot\dfrac{27}{(1+2t)(1-t)^2}\right)-\dfrac{(4-t^2)^2}{9} =$$

$$\frac{t^2(105t^2 - 96t + 32(1 - t^3))}{144(2t + 1)} \ge 0$$

because $105t^2 - 96t + 32(1 - t^3) > 96t^2 - 96t + 32(1 - t^3) = 32(1 - t)^3 > 0.$

Proof of inequality

(**CL**) $\frac{11\sqrt{3}}{5R + 12r} \le \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$

In p,q notation $5R + 12r = \frac{5(p - q)}{4\sqrt{q}} + 12\sqrt{q} = \frac{5p + 43q}{4\sqrt{q}}$ and inequality (**CL**)

becomes $\frac{44\sqrt{3}\sqrt{q}}{5p + 43q} \le \frac{1 + p}{p - q} \iff \frac{44\sqrt{3}}{1 + p} \le \frac{1}{p - q} \bullet \left(43\sqrt{q} + \frac{5p}{\sqrt{q}}\right) = \frac{43q + 5p}{(p - q)\sqrt{q}}$ •.

Since $\frac{5p}{43} > \frac{p^2}{3} \ge q$ then $43\sqrt{q} + \frac{5p}{\sqrt{q}}$ decrease as function of q and, therefore,

decrease $\frac{43q + 5p}{(p - q)\sqrt{q}}$. Hence, $\frac{43q + 5p}{(p - q)\sqrt{q}} - \frac{44\sqrt{3}}{1 + p} \ge \frac{43q_* + 5p}{(p - q_*)\sqrt{q_*}} - \frac{44\sqrt{3}}{1 + p}$

and

$$\frac{(43q_* + 5p)^2}{(p - q_*)^2 q_*} - \frac{5808}{(1 + p)^2} = \frac{\left(43 \bullet \frac{(1 + 2t)(1 - t)^2}{27} + 5 \bullet \frac{1 - t^2}{3}\right)^2}{\left(\frac{1 - t^2}{3} - \frac{(1 + 2t)(1 - t)^2}{27}\right)^2 \frac{(1 + 2t)(1 - t)^2}{27}} - \frac{5808}{\left(1 + \frac{1 - t^2}{3}\right)^2} =$$

$\frac{27t^2(1849t^4 - 15\,052t^3 + 11\,004t^2 + 3872t + 352)}{(2t + 1)(t - 1)^2(t - 2)^2(t + 2)^4} \ge 0$ because

$11\,004t^2 + 3872t + 352 - 15\,052t^3 = 11\,004(t^2 - t^3) + 3872(t - t^3) + 176(1 - t^3) + 176 > 0.$